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Structure Equations in the U_4 Theory of Gravitation

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Abstract Cartan's equations of structure and Bianchi identities in the U_4 theory of gravitation are derived. The relativistic language we adopted is that of Newman-Penrose-Jogia and Griffiths (NPJG). A number of important properties which arise in the development of basic equations are also presented in the NPJG formalism. It is hoped that the essence of non-Riemannian geometry can be summarized by exploiting these equations.

Keywords Newman-Penrose tetrad formalism · Jogia and Griffith formalism · Cartan's equations of structure · Einstein-Cartan theory of gravitation

1 Introduction

The theory of gravitation is known as Einstein's General Theory of Relativity. This theory enables us to understand the mysterious gravitational force through the geometry of the space-time structure. It is described by Riemannian geometry in which the Christoffel symbols are symmetric. Einstein-Cartan theory of gravitation is popularly known as U_4 theory of gravitation. The theory was originated by Cartan [1, 2] by considering the influence of intrinsic spin of matter on the space-time; the same was not included in the Einstein's General Theory of Relativity. The U_4 theory of gravitation is described by the non-Riemannian geometry in which the Christoffel symbols are not symmetric. The non-Riemannian part is described by the affine connections ω^l_{ij} and are defined as

$$\omega^l_{ij} = \Gamma^l_{ij} - K^l_{ij} \tag{1.1}$$

where K_{ijk} is the contortion tensor satisfying

$$K_{i(jk)} = 0, \quad \text{and} \qquad \Gamma^l_{ij} = \Gamma^l_{ji} \tag{1.2}$$

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The covariant derivative of a vector field A_i with respect to the affine connections ω^l_{ij} is defined by

$$A_{i;j} = A_{i/j} + A_l K^l_{ji}, \quad \text{where} \tag{1.3}$$

$$A_{i/j} = A_{i,j} - A_h \Gamma^h_{ji} \tag{1.4}$$

is the covariant derivative with respect to the symmetric Christoffel symbols.

The aim of the paper is to derive the equations of structure and the Bianchi identities in the U_4 theory of gravitation**.** These equations summarized the essence of Riemannian geometry. The techniques of differential forms, which prove to be more powerful and involve enormous computational advantages resulting from the fact that the matrix of the coefficients of the fundamental metric is constant, are exploited to derive these equations. The traditional approach of tensors makes heavy use of Christoffel symbols which are 40 in number and have no invariant significance under the change of co-ordinates in 4 dimensional space-time of general theory of relativity. In differential forms, the role of Christoffel symbols is taken care by connection 1-forms ω_{ij}, which have the property

$$\omega_{ij} = -\omega_{ji} \tag{1.5}$$

so that there are only six components of ω_{ij} in 4-space as opposed to forty Christoffel symbols. These components can be obtained usually very easily from $d\theta^\alpha$ which latter can be simplify the computation of the Riemannian tensor.

Null tetrad of Newman and Penrose [3], basic equations and key entities are presented in Sect. 2. In Sect. 3, the Cartan's equations of structure and Bianchi identities in the U_4 theory of gravitation are delineated in NPJG formalism. A number of important properties which arise in the development of the basic equations are also presented.

2 Null Tetrad

At each point of space-time of U_4 theory of gravitation a tetrad of four null vectors

$$e^i_{(\alpha)} = (l^i, n^i, m^i, \bar{m}^i)$$

is constructed where $l_i, n_i, m_i, \bar{m}_i$ are Newman-Penrose complex orthonormal null vector fields. Dual tetrad of vectors is defined by

$$e^{(\alpha)}_i = (n_i, l_i, -\bar{m}_i, -m_i)$$

such that

$$\eta_{\alpha\beta}(=\eta^{\alpha\beta}) = e^i_{(\alpha)} e_{(\beta)j} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{2.1}$$

and

$$g_{ij} = \eta_{\alpha\beta} e^{(\alpha)}_i e^{(\beta)}_j \tag{2.2}$$

$$\Rightarrow \quad g_{ij} = l_i n_j + n_i l_j - m_i \bar{m}_j - \bar{m}_i m_j \tag{2.3}$$

This relation is commonly known as "completeness relation" for the tetrad. Our basis is complex. Hence a real vector A_i will have complex tetrad components.

$$A_\alpha = A_i e^i_{(\alpha)} \tag{2.4}$$

The condition for the tetrad components of a vector A_i to be real is that

$$A_\alpha \theta^\alpha = \bar{A}_\alpha \bar{\theta}^\alpha,$$

where

$$\theta^\alpha = e^{(\alpha)}_i dx^i \tag{2.5}$$

are the basis 1-forms.

$$\Rightarrow \quad dx^i = \theta^\alpha e^i_{(\alpha)} \tag{2.6}$$

$$\Rightarrow \quad dx^i = \theta^1 l^i + \theta^2 n^i + \theta^3 m^i + \theta^4 \bar{m}^i \tag{2.7}$$

2.1 Complex Basis for Real 2-Forms

Since θ^α form the basis for all 1-forms and that $\theta^\alpha \wedge \theta^\beta$ form a basis for all 2-forms, we denote this basis as

$$\begin{aligned} Z^1 &= \theta^{13} = \theta^1 \wedge \theta^3, \\ Z^2 &= \theta^{12} - \theta^{34} = \theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4, \\ Z^3 &= \theta^{42} = -\theta^2 \wedge \theta^4. \end{aligned} \tag{2.8}$$

Their complex conjugates are defined by

$$\begin{aligned} \bar{Z}^1 &= \theta^{14} = \theta^1 \wedge \theta^4, \\ \bar{Z}^2 &= \theta^{12} + \theta^{34} = \theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4, \\ \bar{Z}^3 &= \theta^{32} = -\theta^2 \wedge \theta^3. \end{aligned} \tag{2.9}$$

Each of these bivectors Z^m and $\bar{Z}^m$ $(m = 1, 2, 3)$ can be used as the basis of all 2-forms. The advantage of using the notation Z^m is that the basis 2-forms $\theta^\alpha \wedge \theta^\beta$ are six in number, while Z^m has essentially 3 components. Each of the bivector Z^m has expansion in tetrad components as

$$Z^1 = \frac{1}{2} Z^1_{\alpha\beta} \theta^\alpha \wedge \theta^\beta,$$

$$Z^1 = 2\delta^1_{[\alpha}\delta^3_{\beta]}.$$

Similarly, we obtain

$$\begin{aligned} Z^2 &= 2(\delta^1_{[\alpha}\delta^2_{\beta]} - \delta^3_{[\alpha}\delta^4_{\beta]}), \quad \text{and} \\ Z^3 &= 2\delta^4_{[\alpha}\delta^2_{\beta]} \end{aligned} \tag{2.10}$$

Each of these bivectors Z^m is self dual satisfying

$$^*Z^m_{\alpha\beta} = i Z^m_{\alpha\beta} \tag{2.11}$$

and their complex conjugates $\bar{Z}^m$ are anti-self dual satisfying

$$^*\bar{Z}^m_{\alpha\beta} = -i\bar{Z}^m_{\alpha\beta} \tag{2.12}$$

Tensor components of any bevector Z^m can be obtained by expressing it as a linear combination of its tetrad components. Then we have

$$\begin{aligned} Z^1_{hi} &= Z^1_{\alpha\beta} e^{(\alpha)}_h e^{(\beta)}_i \\ &= 2\delta^1_{[\alpha}\delta^3_{\beta]} e^{(\alpha)}_h e^{(\beta)}_i \\ \Rightarrow \quad Z^1_{hi} &= -2n_{[h}\bar{m}_{i]}. \end{aligned} \tag{2.13}$$

Similarly, we obtain

$$Z^2_{hi} = -2[l_{[h}n_{i]} - m_{[h}\bar{m}_{i]}], \tag{2.14}$$

and

$$Z^3_{hi} = 2l_{[h}m_{i]} \tag{2.15}$$

where

$$\frac{1}{2} Z^m_{ij} Z^{nij} = \gamma^{mn}$$

and

$$\gamma^{mn} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{2.16}$$

Its inverse matrix γ_{mn} is obtain from

$$\gamma^{mn}\gamma_{mp} = \delta^n_p$$

and is therefore given by

$$\gamma_{mn} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{2} & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{2.17}$$

3 Equations of Structure in Einstein-Cartan Theory of Gravitation

The essence of Riemannian geometry is summarized in the equations of structure. These equations relate the exterior differentials $d\theta^\alpha$ to connection 1-forms ω_{ij} and $d\omega_{ij}$ to curvature 2-forms Ω_{ij}. To derive these equations in the U_4 theory of gravitation we start with the definition of Ricci's coefficient of rotations as

$$\gamma_{\alpha\beta\gamma} = -e_{(\alpha)i;j} e^i_{(\beta)} e^j_{(\gamma)} \tag{3.1}$$

Using (1.3) we obtain

$$\gamma_{\alpha\beta\gamma} = \gamma^0_{\alpha\beta\gamma} + K_{\gamma\alpha\beta} \tag{3.2}$$

where

$$K_{\alpha\beta\gamma} = K_{ijk} e^i_{(\alpha)} e^j_{(\beta)} e^k_{(\gamma)} \tag{3.3}$$

are the tetrad components of the contortion tensor satisfying

$$K_{\alpha(\beta\gamma)} = 0.$$

Here and in the following the over head zero indicates the quantities in Einstein theory of gravitation.

Now taking the exterior derivative of the basis 1-forms θ^α we obtain

$$\begin{aligned} d\theta^\alpha &= e^{(\alpha)}_{i;j} dx^j \wedge dx^i \\ \Rightarrow \quad d\theta^\alpha &= \gamma^\alpha_{\beta\gamma} \theta^\beta \wedge \theta^\gamma \end{aligned} \tag{3.4}$$

Consequently, using (3.2) we get

$$\begin{aligned} d\theta^\alpha &= -\gamma^{0\alpha}_{\beta\gamma} \theta^\gamma \wedge \theta^\beta + K^\alpha_{\beta\gamma} \theta^\beta \wedge \theta^\gamma \quad \text{or} \\ d\theta^\alpha &= -\omega^{0\alpha}_\beta \wedge \theta^\beta + T^\alpha \end{aligned} \tag{3.5}$$

where

$$\omega^{0\alpha}_\beta = \gamma^{0\alpha}_{\beta\gamma} \theta^\gamma \tag{3.6}$$

are components of connection 1-forms and

$$T^\alpha = K^\alpha_{\beta\gamma} \theta^\beta \wedge \theta^\gamma \tag{3.7}$$

are 2-forms. Equation (3.5) is the Cartan's first equation of structure in the U_4 theory of gravitation.

If $\Omega_{\alpha\beta}$ are the tetrad components of the curvature 2-form in U_4 theory, then Cartan's second equation of structure becomes

$$\Omega_{\alpha\beta} = \frac{1}{2} R_{\alpha\beta\gamma\delta} \theta^\gamma \wedge \theta^\delta = \Omega^0_{\alpha\beta} + K_{\varepsilon\alpha\beta} T^\varepsilon + \left[dK_{\varepsilon\alpha\beta} + \eta^{\sigma\rho} \left\{ \begin{matrix} K_{\varepsilon\sigma\beta} \omega^0_{\alpha\rho} - K_{\sigma\alpha\beta} \omega^0_{\rho\varepsilon} - \\ -K_{\varepsilon\alpha\rho} \omega^0_{\sigma\beta} - K_{\varepsilon\alpha\rho} K_{\delta\sigma\beta} \theta^\delta \end{matrix} \right\} \right] \wedge \theta^\varepsilon \tag{3.8}$$

where

$$\Omega^0_{\alpha\beta} = d\omega^0_{\alpha\beta} + \eta^{\sigma\rho} \omega^0_{\alpha\sigma} \wedge \omega^0_{\rho\beta} \tag{3.9}$$

is the Cartan's second equation of structure in Einstein's theory of gravitation. The contortion components $K_{\alpha\beta\gamma}$ are the quantities by which the curvature 2-forms differ from their values in a Riemannian space-time, and

$$\omega_{\alpha\beta} = \omega^0_{\alpha\beta} + K_{\gamma\alpha\beta} \theta^\gamma \tag{3.10}$$

3.1 Newman-Penrose-Jogia-Griffiths Concomitants of Equations of Structure

Analogous to the 'amazingly useful Newman-Penrose [3] formalism for Einstein's theory of gravitation, Jogia and Griffiths [4] have developed a null formalism for studying U_4 theory of gravitation. We adopt Jogia-Griffiths formalism for the description of equations of structure in the U_4 theory of gravitation. We refer the formalism as Newman-Penrose-Jogia-Griffiths (NPJG) formalism for the U_4 theory of gravitation.

In terms of the connection 1-forms, the first equations of structure are

$$\begin{aligned} d\theta^1 &= -(\sigma_2^0 + \bar{\sigma}_2^0) \wedge \theta^1 - \bar{\sigma}_3^0 \wedge \theta^3 - \sigma_3^0 \wedge \theta^4 + T^1, \\ d\theta^2 &= (\sigma_2^0 + \bar{\sigma}_2^0) \wedge \theta^2 + \sigma_1^0 \wedge \theta^3 + \bar{\sigma}_1^0 \wedge \theta^3 + T^2, \\ d\theta^3 &= \bar{\sigma}_1^0 \wedge \theta^1 - \sigma_3^0 \wedge \theta^2 - (\sigma_2^0 - \bar{\sigma}_2^0) \wedge \theta^3 + T^3, \\ d\theta^4 &= \sigma_1^0 \wedge \theta^1 - \bar{\sigma}_3^0 \wedge \theta^2 + (\sigma_2^0 - \bar{\sigma}_2^0) \wedge \theta^4 + T^4, \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} \sigma_1 &= -\omega_{13} = \sigma_1^0 + \sigma_1^1, \\ \sigma_1 &= -\frac{1}{2}(\omega_{12} - \omega_{34}) = \sigma_2^0 + \sigma_2^1, \\ \sigma_3 &= \omega_{24} = \sigma_3^0 + \sigma_3^1. \end{aligned} \tag{3.12}$$

The contravariant suffix 1 indicates that these terms arise due to intrinsic spin of matter in the U_4 theory of gravitation and are defined by

$$\begin{aligned} \sigma_1^1 &= \kappa_1\theta^1 + \tau_1\theta^2 + \sigma_1^*\theta^3 + \rho_1\theta^4, \\ \bar{\sigma}_1^1 &= \bar{\kappa}_1\theta^1 + \bar{\tau}_1\theta^2 + \bar{\rho}_1\theta^3 + \bar{\sigma}_1^*\theta^4, \\ \sigma_2^1 &= \varepsilon_1\theta^1 + \gamma_1\theta^2 + \beta_1\theta^3 + \alpha_1\theta^4, \\ \bar{\sigma}_2^1 &= \bar{\varepsilon}_1\theta^1 + \bar{\gamma}_1\theta^2 + \bar{\alpha}_1\theta^3 + \bar{\beta}_1\theta^4, \\ \sigma_3^1 &= \pi_1\theta^1 + \nu_1\theta^2 + \mu_1\theta^3 + \lambda_1\theta^4, \\ \bar{\sigma}_3^1 &= \bar{\pi}_1\theta^1 + \bar{\nu}_1\theta^2 + \bar{\mu}_1\theta^4 + \bar{\lambda}_1\theta^3, \end{aligned} \tag{3.13a}$$

and

$$\begin{aligned} \sigma_1^0 &= -\omega_{13}^0 = \kappa^0\theta^1 + \tau^0\theta^2 + \sigma^{*0}\theta^3 + \rho^0\theta^4, \\ \sigma_2^0 &= -\frac{1}{2}(\omega_{12}^0 - \omega_{34}^0) = \varepsilon^0\theta^1 + \gamma^0\theta^2 + \beta^0\theta^3 + \alpha^0\theta^4, \\ \sigma_3^0 &= \omega_{24}^0 = \pi^0\theta^1 + \nu^0\theta^2 + \mu^0\theta^3 + \lambda^0\theta^4, \end{aligned} \tag{3.13b}$$

and

$$\begin{aligned} T^1 &= (\gamma_1 + \bar{\gamma}_1)\theta^{12} + (\bar{\alpha}_1 + \beta_1 - \bar{\pi}_1)\theta^{13} + (\alpha_1 + \bar{\beta}_1 - \pi_1)\theta^{14} - \bar{\nu}_1\theta^{23} \\ &\quad - \nu_1\theta^{24} - (\mu_1 - \mu_1)\theta^{34}, \\ T^2 &= (\varepsilon_1 + \bar{\varepsilon}_1)\theta^{12} + \kappa_1\theta^{13} + \bar{\kappa}_1\theta^{14} + (\tau_1 - \bar{\alpha}_1 - \beta_1)\theta^{23} \\ &\quad + (\bar{\tau}_1 - \alpha_1 - \bar{\beta}_1)\theta^{24} - (\rho_1 - \bar{\rho}_1)\theta^{34}, \\ T^3 &= -(\bar{\tau}_1 + \pi_1)\theta^{12} + (\bar{\varepsilon}_1 - \varepsilon_1 - \bar{\rho}_1)\theta^{13} + \bar{\sigma}_1^*\theta^{14} + (\mu_1 - \gamma_1 + \bar{\gamma}_1)\theta^{23} \\ &\quad + \lambda_1\theta^{24} + (\alpha_1 - \bar{\beta}_1)\theta^{34}, \\ T^4 &= -(\tau_1 + \bar{\pi}_1)\theta^{12} - \sigma_1^*\theta^{13} + (\varepsilon_1 - \bar{\varepsilon}_1 - \rho_1)\theta^{14} + \bar{\lambda}_1\theta^{23} \\ &\quad + (\bar{\mu}_1 - \bar{\gamma}_1 + \gamma_1)\theta^{24} + (\bar{\alpha}_1 - \beta_1)\theta^{34}, \end{aligned} \tag{3.14}$$

or equivalently,

$$\begin{aligned}
d\theta^1 &= (\gamma^0 + \gamma_1 + \bar{\gamma}^0 + \bar{\gamma}_1)\theta^{12} + (\bar{\alpha}^0 + \bar{\alpha}_1 + \beta^0 + \beta_1 - \bar{\pi}^0 - \bar{\pi}_1)\theta^{13} \\
&\quad + (\alpha^0 + \alpha_1 + \bar{\beta}^0 + \bar{\beta}_1 - \pi^0 - \pi_1)\theta^{14} - (\bar{\nu}^0 + \bar{\nu}_1)\theta^{23} - (\nu^0 + \nu_1)\theta^{24} \\
&\quad - (\mu^0 + \mu_1 - \bar{\mu}^0 - \bar{\mu}_1)\theta^{34} \\
d\theta^2 &= (\varepsilon^0 + \varepsilon_1 + \bar{\varepsilon}^0 + \bar{\varepsilon}_1)\theta^{12} + (\kappa^0 + \kappa_1)\theta^{13} + (\bar{\kappa}^0 + \bar{\kappa}_1)\theta^{14} \\
&\quad + (\tau^0 + \tau_1 - \bar{\alpha}^0 - \bar{\alpha}_1 - \beta^0 - \beta_1)\theta^{23} + (\bar{\tau}^0 + \bar{\tau}_1 - \alpha^0 - \alpha_1 - \bar{\beta}^0 - \bar{\beta}_1)\theta^{24} \\
&\quad - (\rho^0 + \rho_1 - \bar{\rho}^0 - \bar{\rho}_1)\theta^{34} \\
d\theta^3 &= -(\pi^0 + \pi_1 + \bar{\tau}^0 + \bar{\tau}_1)\theta^{12} - (\bar{\rho}^0 + \bar{\rho}_1 + \varepsilon^0 + \varepsilon_1 - \bar{\varepsilon}^0 - \bar{\varepsilon}_1)\theta^{13} - (\bar{\sigma}^{*0} + \bar{\sigma}_1^*)\theta^{14} \\
&\quad + (\mu^0 + \mu_1 - \gamma^0 - \gamma_1 + \bar{\gamma}^0 + \bar{\gamma}_1)\theta^{23} + (\lambda^0 + \lambda_1)\theta^{24} + (\alpha^0 + \alpha_1 - \bar{\beta}^0 - \bar{\beta}_1)\theta^{34} \\
d\theta^4 &= -(\bar{\pi}^0 + \bar{\pi}_1 + \tau^0 + \tau_1)\theta^{12} - (\sigma^{*0} + \sigma_1^*)\theta^{13} - (\rho^0 + \rho_1 + \bar{\varepsilon}^0 + \bar{\varepsilon}_1 - \varepsilon^0 - \varepsilon_1)\theta^{14} \\
&\quad + (\bar{\lambda}^0 + \bar{\lambda}_1)\theta^{23} + (\bar{\mu}^0 + \bar{\mu}_1 - \bar{\gamma}^0 - \bar{\gamma}_1 + \gamma^0 + \gamma_1)\theta^{24} - (\bar{\alpha}^0 + \bar{\alpha}_1 - \beta^0 - \beta_1)\theta^{34}.
\end{aligned} \tag{3.15}$$

The exterior derivatives of bivectors are

$$\begin{aligned}
dZ^1 &= -2\sigma_2^0 \wedge Z^1 - \sigma_3^0 \wedge Z^2 - (\mu_1 - 2\gamma_1)\theta^{123} - \lambda_1\theta^{124} + (\pi_1 - 2\alpha_1)\theta^{134} + \nu_1\theta^{234}, \\
dZ^2 &= 2\sigma_1^0 \wedge Z^1 - 2\sigma_3^0 \wedge Z^3 - 2\tau_1\theta^{123} + 2\pi_1\theta^{124} + 2\rho_1\theta^{134} - 2\mu_1\theta^{234}, \\
dZ^3 &= \sigma_1^0 \wedge Z^2 + 2\sigma_2^0 \wedge Z^3 + \sigma_1^*\theta^{123} + (\rho_1 - 2\varepsilon_1)\theta^{124} - \kappa_1\theta^{134} - (\tau_1 - 2\beta_1)\theta^{234}
\end{aligned} \tag{3.16}$$

Equivalently, we also write these equations in the form

$$\begin{aligned}
dZ^1 &= -(\mu^0 + \mu_1 - 2\gamma^0 - 2\gamma_1)\theta^{123} - (\lambda^0 + \lambda_1)\theta^{124} + (\pi^0 + \pi_1 - 2\alpha^0 - 2\alpha_1)\theta^{134} \\
&\quad + (\nu^0 + \nu_1)\theta^{234}, \\
dZ^2 &= -2(\tau^0 + \tau_1)\theta^{123} + 2(\pi^0 + \pi_1)\theta^{124} + 2(\rho^0 + \rho_1)\theta^{134} - 2(\mu^0 + \mu_1)\theta^{234}, \\
dZ^3 &= (\sigma^{*0} + \sigma_1^*)\theta^{123} + (\rho^0 + \rho_1 - 2\varepsilon^0 - 2\varepsilon_1)\theta^{124} - (\kappa^0 + \kappa_1)\theta^{134} \\
&\quad - (\tau^0 + \tau_1 - 2\beta^0 - 2\beta_1)\theta^{234}.
\end{aligned} \tag{3.17}$$

The differentials of the basis three forms are similarly obtained as

$$\begin{aligned}
d\theta^{123} &= -(\sigma_2^0 - \bar{\sigma}_2^0) \wedge \theta^{123} - \bar{\sigma}_1^0 \wedge \theta^{134} - \sigma_3^0 \wedge \theta^{234} + (\bar{\tau}_1 - \pi_1 + \alpha_1 - \bar{\beta}_1)\theta^{1234}, \\
d\theta^{124} &= (\sigma_2^0 - \bar{\sigma}_2^0) \wedge \theta^{124} + \sigma_1^0 \wedge \theta^{134} + \bar{\sigma}_3^0 \wedge \theta^{234} - (\bar{\alpha}_1 - \beta_1 + \tau_1 - \bar{\pi}_1)\theta^{1234}, \\
d\theta^{134} &= \bar{\sigma}_3^0 \wedge \theta^{123} - \sigma_3^0 \wedge \theta^{124} - (\sigma_2^0 + \bar{\sigma}_2^0) \wedge \theta^{134} + (\gamma_1 + \bar{\gamma}_1 - \mu_1 - \bar{\mu}_1)\theta^{1234}, \\
d\theta^{234} &= \sigma_1^0 \wedge \theta^{123} - \bar{\sigma}_1^0 \wedge \theta^{124} + (\sigma_2^0 + \bar{\sigma}_2^0) \wedge \theta^{234} + (\varepsilon_1 + \bar{\varepsilon}_1 - \rho_1 - \bar{\rho}_1)\theta^{1234}
\end{aligned} \tag{3.18}$$

From these equations we obtain

$$\begin{aligned}
&\theta^1 \wedge dZ^1 = (\nu^0 + \nu_1)\theta^{1234}, \qquad \theta^2 \wedge dZ^1 = (2\alpha^0 + 2\alpha_1 - \pi^0 - \pi_1)\theta^{1234}, \\
&\theta^3 \wedge dZ^1 = -(\lambda^0 + \lambda_1)\theta^{1234}, \qquad \theta^4 \wedge dZ^1 = (\mu^0 + \mu_1 - 2\gamma^0 - 2\gamma_1)\theta^{1234},
\end{aligned}$$

$$\theta^1 \wedge dZ^2 = -2(\mu^0 + \mu_1)\theta^{1234}, \qquad \theta^2 \wedge dZ^2 = -2(\rho^0 + \rho_1)\theta^{1234},$$
$$\theta^3 \wedge dZ^2 = 2(\pi^0 + \pi_1)\theta^{1234}, \qquad \theta^4 \wedge dZ^2 = 2(\tau^0 + \tau_1)\theta^{1234},$$
$$\theta^1 \wedge dZ^3 = (\bar{\tau}^0 + \bar{\tau}_1 - 2\bar{\beta}^0 - 2\bar{\beta}_1)\theta^{1234}, \qquad \theta^2 \wedge dZ^3 = -(\bar{\kappa}^0 + \bar{\kappa}_1)\theta^{1234},$$
$$\theta^3 \wedge dZ^3 = (\bar{\sigma}^{*0} + \bar{\sigma}_1^*)\theta^{1234}, \qquad \theta^4 \wedge dZ^3 = -(\bar{\rho}^0 + \bar{\rho}_1 - 2\bar{\varepsilon}^0 - 2\bar{\varepsilon}_1)\theta^{1234}.$$

3.2 Cartan's Second Equation of the Structure

We shall now derive the second equation of structure which connects the curvature 2-form to the exterior differentials of the connection 1-forms. The second equation of structure, in terms of the components become

$$\begin{aligned}
\Sigma_1 &= \Sigma_1^0 + d\sigma_1^1 + 2\sigma_1^0 \wedge \sigma_2^1 - 2\sigma_2^0 \wedge \sigma_1^1 + 2\sigma_1^1 \wedge \sigma_2^1, \\
\Sigma_2 &= \Sigma_2^0 + d\sigma_2^1 + \sigma_1^0 \wedge \sigma_3^1 - \sigma_3^0 \wedge \sigma_1^1 + \sigma_1^1 \wedge \sigma_3^1, \\
\Sigma_3 &= \Sigma_3^0 + d\sigma_3^1 + 2\sigma_2^0 \wedge \sigma_3^1 - 2\sigma_3^0 \wedge \sigma_2^1 + 2\sigma_2^1 \wedge \sigma_3^1,
\end{aligned} \tag{3.19}$$

where we have denoted

$$\Sigma_1 = -\Omega_{13}, \qquad \Sigma_2 = -\frac{1}{2}(\Omega_{12} - \Omega_{34}), \quad \text{and} \quad \Sigma_3 = \Omega_{24} \tag{3.20}$$

and

$$\begin{aligned}
\Sigma_1^0 &= -\Omega_{13}^0 = d\sigma_1^0 + 2\sigma_1^0 \wedge \sigma_2^0, \\
\Sigma_2^0 &= -\frac{1}{2}(\Omega_{12}^0 - \Omega_{34}^0) = d\sigma_2^0 + \sigma_1^0 \wedge \sigma_3^0, \\
\Sigma_3^0 &= \Omega_{24}^0 = d\sigma_3^0 + 2\sigma_2^0 \wedge \sigma_3^0,
\end{aligned} \tag{3.21}$$

are the components of second equations of structure in Einstein gravitation theory [5].

Theorem *If $\Omega_{\alpha\beta}^0$ are components of curvature 2-forms in Einstein's theory of gravitation and T^α are 2-forms, then*

$$\Omega_{\alpha\beta}^0 \wedge T^\alpha = 0.$$

Proof From Cartan's first equation of structure in the U_4 theory of gravitation (3.5) we have

$$T^\alpha = d\theta^\alpha + \omega_\beta^{0\alpha} \wedge \theta^\beta$$

Taking exterior derivative of this equation we obtain

$$dT^\alpha = d\omega_\beta^{0\alpha} \wedge \theta^\beta - \omega_\beta^{0\alpha} \wedge d\theta^\beta \tag{3.22}$$

Eliminating the term $d\omega_\beta^{0\alpha}$ by using Cartan's second equations of structure in Einstein's gravitation theory (3.8) we get

$$dT^\alpha = \Omega_\beta^{0\alpha} \wedge \theta^\beta - \omega_\beta^{0\alpha} \wedge T^\beta,$$

where $\Omega_\beta^{0\alpha} \wedge \theta^\beta = 0$.

$$\Rightarrow \quad dT^\alpha = -\omega_\beta^{0\alpha} \wedge T^\beta. \tag{3.23}$$

Taking exterior derivative of the above equation we get

$$d\omega^{0\alpha}_{\beta} \wedge T^{\beta} - \omega^{0\alpha}_{\beta} \wedge dT^{\beta} = 0. \tag{3.24}$$

Eliminating $d\omega^{0\alpha}_{\beta}$ by using Cartan's second equations of structure (3.8) and using (3.22) we obtain

$$\Omega^{0}_{\alpha\beta} \wedge T^{\alpha} = 0.$$

However, note that

$$\Omega_{\alpha\beta} \wedge T^{\alpha} \neq 0,$$

but

$$\Omega_{\alpha\beta} \wedge T^{\alpha} = [dK_{\gamma\alpha\beta} - K_{\sigma\alpha\beta}\omega^{0\sigma}_{\gamma} + K_{\gamma\sigma\beta}\omega^{0\sigma}_{\alpha} - K_{\gamma\alpha\sigma}\omega^{0\sigma}_{\beta} - K^{\sigma}_{\gamma\alpha}K_{\rho\sigma\beta}\theta^{\rho}] \wedge \theta^{\gamma} \wedge T^{\beta} + K_{\gamma\alpha\beta}T^{\gamma} \wedge T^{\beta}.$$

□

3.3 Bianchi Identities

In Einstein-Cartan theory of gravitation, the Bianchi identities are obtain by taking the exterior derivative of (3.19), we obtain

$$d\Sigma_1 = d\Sigma^0_1 + 2(d\sigma^0_1 \wedge \sigma^1_2 - \sigma^0_1 \wedge d\sigma^1_2) - 2(d\sigma^0_2 \wedge \sigma^1_1 - \sigma^0_2 \wedge d\sigma^1_1) + 2(d\sigma^1_1 \wedge \sigma^1_2 - \sigma^1_1 \wedge d\sigma^1_2) \tag{3.25}$$

Using (3.18) and (3.10), we obtain after simplifying (3.25) as

$$d\Sigma_1 = 2\Sigma^0_1 \wedge \sigma^1_2 - 2\Sigma_2 \wedge \sigma^0_1 - \Sigma^0_1 \wedge \sigma^1_1 + 2\Sigma_1 \wedge \sigma^0_2 + 2\sigma^0_1 \wedge \sigma^1_1 \wedge \sigma^1_3 - 4\sigma^0_2 \wedge \sigma^1_1 \wedge \sigma^1_2.$$

Similarly, we obtain

$$d\Sigma_2 = \Sigma_1 \wedge \sigma^0_3 + \Sigma_1 \wedge \sigma^1_3 - \Sigma_3 \wedge \sigma^0_1 - \Sigma_3 \wedge \sigma^1_1 + 2\sigma^0_1 \wedge \sigma^1_1 \wedge \sigma^1_3 - 2\sigma^0_1 \wedge \sigma^1_2 \wedge \sigma^1_3, \tag{3.26}$$

and

$$d\Sigma_3 = 2\Sigma_2 \wedge \sigma^0_2 - 2\Sigma_3 \wedge \sigma^0_2 + 2\Sigma_2 \wedge \sigma^1_3 - 2\Sigma_3 \wedge \sigma^1_3,$$

In terms of bivector basis $(Z^m, \bar{Z}^m)$, the components of curvature 2-forms are given by

$$\Sigma_m = \left(C_{mn} + \frac{R}{12}\gamma_{mn}\right)Z^n + E_{mn}\bar{Z}^n, \quad m, n = 1, 2, 3, \tag{3.27}$$

where

$$C_{mn} = \begin{pmatrix} \psi_0 & \psi_1 - \phi_0 & \psi_2 - \phi_1 \\ \psi_1 + \phi_0 & \psi_2 - \chi & \psi_3 - \phi_2 \\ \psi_2 + \phi_1 & \psi_3 + \phi_2 & \psi_4 \end{pmatrix} \tag{3.28}$$

and

$$E_{mn} = \begin{pmatrix} \phi_{00} - i\Theta_{00} & \phi_{01} - i\Theta_{01} & \phi_{02} - i\Theta_{02} \\ \phi_{10} - i\Theta_{10} & \phi_{11} - i\Theta_{11} & \phi_{12} - i\Theta_{12} \\ \phi_{20} - i\Theta_{20} & \phi_{21} - i\Theta_{21} & \phi_{22} - i\Theta_{22} \end{pmatrix} \tag{3.29}$$

where ϕ_{AB} $(A, B = 0, 1, 2)$ are the familiar nine components of a Hermitian 3×3 matrix, ϕ_A the three complex components and R the Ricci scalar together determine the 16 independent components of Ricci tensor in U_4 theory of gravitation. Also the five complex tetrad components of the Weyl tensor, Θ_{AB} are nine components of a Hermitian matrix and χ is a real scalar together determine the 20 independent components of the trace-free part of the curvature tensor. All these quantities are defined in [4].

Now using the relation $\Omega_{\alpha\beta} = \frac{1}{2} R_{\alpha\beta\gamma\delta}\theta^\gamma \wedge \theta^\delta$ and (3.27), we readily obtain the 36 independent components of curvature tensor as

$$\begin{aligned}
&R_{1212} = -\psi_2 - \bar{\psi}_2 - 2\phi_{11} + 2\Lambda, && R_{1213} = -\psi_1 - i\Theta_{01} - \phi_{01} + \phi_0, \\
&R_{1214} = -\bar{\psi}_1 + i\Theta_{10} - \phi_{10} - \bar{\phi}_0, && R_{1223} = \bar{\psi}_3 - i\Theta_{12} - \bar{\phi}_2 + \phi_{12}, \\
&R_{1224} = \psi_3 + i\Theta_{21} + \phi_{21} - \phi_2, && R_{1234} = \psi_2 - \bar{\psi}_2 + 2i\Theta_{11} - 2i\chi, \\
&R_{1312} = -\psi_1 + i\Theta_{01} - \phi_{01} + \phi_0, && R_{1313} = -\psi_0, \\
&R_{1314} = i\Theta_{00} - \phi_{00}, && R_{1323} = -i\Theta_{02} + \phi_{02}, \\
&R_{1324} = \psi_2 - \phi_1 + 2\Lambda, && R_{1334} = \psi_1 + i\Theta_{01} - \phi_{01} - \phi_0, \\
&R_{1412} = -\bar{\psi}_1 - i\Theta_{10} - \phi_{10} + \bar{\phi}_0, && R_{1413} = -i\Theta_{00} - \phi_{00}, \\
&R_{1414} = -\bar{\psi}_0, && R_{1423} = \bar{\psi}_2 - \bar{\phi}_1 + 2\Lambda, \\
&R_{1424} = i\Theta_{20} + \phi_{20}, && R_{1434} = -\bar{\psi}_1 + i\Theta_{10} + \phi_{10} + \bar{\phi}_0, \\
&R_{2312} = \bar{\psi}_3 + i\Theta_{12} + \phi_{12} + \bar{\phi}_2, && R_{2313} = i\Theta_{02} + \phi_{02}, \\
&R_{2314} = \bar{\psi}_2 + \bar{\phi}_1 + 2\Lambda, && R_{2323} = -\bar{\psi}_4, \\
&R_{2324} = -i\Theta_{22} - \phi_{22}, && R_{2334} = \bar{\psi}_3 - i\Theta_{12} - \phi_{12} + \bar{\phi}_2, \\
&R_{2412} = \psi_3 - i\Theta_{21} + \phi_{21} + \phi_2, && R_{2413} = \psi_2 + \phi_1 + 2\Lambda, \\
&R_{2414} = -i\Theta_{20} + \phi_{20}, && R_{2423} = i\Theta_{22} - \phi_{22}, \\
&R_{2424} = -\psi_4, && R_{2434} = -\psi_3 - i\Theta_{21} + \phi_{21} - \phi_2, \\
&R_{3412} = \psi_2 - \bar{\psi}_2 - 2i\Theta_{11} - 2i\chi, && R_{3413} = \psi_1 - i\Theta_{01} - \phi_{01} + \phi_0, \\
&R_{3414} = -\bar{\psi}_1 - i\Theta_{10} + \phi_{10} - \bar{\phi}_0, && R_{3423} = \bar{\psi}_3 + i\Theta_{12} - \phi_{12} - \bar{\phi}_2, \\
&R_{3424} = -\psi_3 + i\Theta_{21} + \phi_{21} + \phi_2, && R_{3434} = -\psi_2 - \bar{\psi}_2 + 2\phi_{11} + 2\Lambda.
\end{aligned} \tag{3.30}$$

The trace free part of the Weyl tensor has components

$$
\begin{aligned}
&C_{1212}=C_{3434}=(-\psi_2+i\chi-i\Theta_{11})+c.c. && C_{1213}=-C_{1234}=-\psi_1-i\Theta_{01},\\
&C_{1214}=C_{1434}=-\bar{\psi}_1+i\Theta_{10}, && C_{1223}=C_{2334}=\bar{\psi}_3-i\Theta_{12},\\
&C_{1224}=-C_{2434}=\psi_3+i\Theta_{21}, && C_{1234}=\psi_2-\bar{\psi}_2+2i\Theta_{11}-2i\chi,\\
&C_{1312}=-C_{3413}=-\psi_1+i\Theta_{01}, && C_{1313}=-\psi_0,\\
&C_{1314}=-C_{1413}=i\Theta_{00}, && C_{1323}=-C_{2313}=-i\Theta_{02},\\
&C_{1324}=C_{2413}=\psi_2, && C_{1412}=C_{3414}=-\bar{\psi}_1-i\Theta_{10},\\
&C_{1414}=-\bar{\psi}_0, && C_{1423}=C_{2314}=\bar{\psi}_2,\\
&C_{1424}=-C_{2414}=i\Theta_{20}, && C_{2312}=C_{3423}=\bar{\psi}_3+i\Theta_{12},\\
&C_{2323}=-\bar{\psi}_4, && C_{2324}=-C_{2423}=-i\Theta_{22},\\
&C_{2412}=-C_{3424}=\psi_3-i\Theta_{21}, && C_{2424}=-\psi_4,\\
&C_{3412}=\psi_2-\bar{\psi}_2-2i\Theta_{11}-2i\chi.
\end{aligned}
\tag{3.31}
$$

The Weyl tensor C_{hijk} in tetrad components is expressed as

$$
\begin{aligned}
C_{hijk}=&[-\psi_4 Z^1_{hi}Z^1_{jk}+\psi_3(Z^1_{hi}Z^2_{jk}+Z^2_{hi}Z^1_{jk})-\psi_2(Z^1_{hi}Z^3_{jk}+Z^2_{hi}Z^2_{jk}+Z^3_{hi}Z^1_{jk})\\
&+\psi_1(Z^2_{hi}Z^3_{jk}+Z^3_{hi}Z^2_{jk})-\psi_0 Z^3_{hi}Z^3_{jk}+i\Theta_{00}Z^3_{hi}\bar{Z}^3_{jk}+i\Theta_{01}(\bar{Z}^2_{hi}Z^3_{jk}-Z^3_{hi}\bar{Z}^2_{jk})\\
&+i\Theta_{02}(\bar{Z}^1_{hi}Z^3_{jk}+Z^3_{hi}\bar{Z}^1_{jk})+i\Theta_{11}Z^2_{hi}\bar{Z}^2_{jk}-i\Theta_{12}(Z^2_{hi}\bar{Z}^1_{jk}-\bar{Z}^1_{hi}Z^2_{jk})\\
&+i\Theta_{22}\bar{Z}^1_{hi}Z^1_{jk}+2i\chi Z^2_{hi}Z^2_{jk}]+[c.c.]
\end{aligned}
\tag{3.32}
$$

where c.c. denotes the complex conjugate of earlier term.

4 Conclusion

The Cartan's equations of structure, the Einstein field equations and the Bianchi identities are derived in Einstein Cartan theory of gravitation by using the technique of differential forms. These equations are transcribed in the NPJG formalism. It is hoped that the essence of non-Riemannian geometry can be summarized by exploiting these equations. The expression for the Weyl Curvature tensor is also derived explicitly. It is worth to note that when the components of contortion tensor are zero the results reduce to the results of Einstein theory of gravitation.

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